

EFFICIENT ESTIMATION OF NEGATIVE
BINOMIAL PARAMETERS USING
EMPIRICAL LA PLACE TRANSFORM

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THESIS

EFFICIENT ESTIMATION OF NEGATIVE BINOMIAL
PARAMETERS USING EMPIRICAL LA PLACE TRANSFORM

by

Resai Caglayan

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Efficient Estimation of Negative Binomial
Parameters Using Empirical La Place Transform

by

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requirements for the degree of

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ABSTRACT

A new method, based on empirical La Place transform, was developed to find asymptotically efficient estimates of negative binomial distribution parameters. These estimates were found fairly close to those found by the method of maximum likelihood. Efficiencies over 95 percent were obtained. The method was tested with a set of data, generated by computer, and found to be satisfactory except in a few cases. Maximum likelihood also fails to be satisfactory in these cases.

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I. INTRODUCTION

The need for readily computed parameter estimates is great. Maximum likelihood estimators are known to be asymptotically efficient, but in many settings they are hard to find. The most popular alternative is the method of moments which usually yields readily computed estimates, but sometimes these estimates are not very efficient. See ref. [2]. This study looks at the efficiency of a method which uses the probability generating function (empirical La Place transform evaluated at a desirable value of its argument, u_0).

The method presented herein requires computing power greater than that of the method of moments, but less than that of maximum likelihood, which calls for a psi function capability. It can be used with a hand held calculator that has the lower order transcendental functions, i.e., logarithm, roots and power.

The basic idea is to select a system of estimating equations which equates various statistics to their expected values. In chapter II, estimating equations were set and efficiency computations were done using theoretical work presented in [3]. Chapter III contains the procedure to be followed, applications of the method to some data sets, and comparisons with the other methods.

In chapter IV, efficiency and optimum u_0 and t_0 ($= -\log u_0$) tables for various p-r combinations are given and efficiency contours in three different planes are graphed.

II. METHODOLOGY

A negative binomial random variable X has the probability function

$$f(X=x; r, p) = \frac{\Gamma(r+x)}{x! \Gamma(r)} q^x p^r \quad (2.1)$$

for $x = 0, 1, 2, \dots$ $0 < r$, $0 < p < 1$, $p + q = 1$. The partial derivatives of its logarithm are, using $h = \log f$

$$\frac{\partial h}{\partial p} = \frac{r}{p} - \frac{x}{q}$$

$$\frac{\partial h}{\partial r} = \Psi(r+x) - \Psi(r) + \log p$$

Using the basic recursive formula for the psi function [1]

$$\Psi(r+x) - \Psi(r) = \sum_{j=1}^x \frac{1}{r-1+j}$$

one may then express the system of maximum likelihood equations as

$$\bar{x} - \frac{rq}{p} = 0 \quad (2.3)$$

$$\text{Ave}_{i=1, \dots, n} \sum \frac{1}{r-1+j} + \log p = 0$$

where the values x_1, \dots, x_n are the data that result when the population (2.1) is sampled n times. The system (2.3) is nonlinear in r, p and difficult to solve. Iterative methods must be used and the second member of (2.3) (or some version thereof) must be recomputed in each cycle. This is the main computational difficulty in using maximum likelihood in this setting.

The information matrix Λ , which is defined as:

$$\Lambda = -E \begin{bmatrix} \frac{\partial^2 h}{\partial p^2} & \frac{\partial^2 h}{\partial p \partial r} \\ \frac{\partial^2 h}{\partial r \partial p} & \frac{\partial^2 h}{\partial r^2} \end{bmatrix}$$

is needed for efficiency calculations and can be developed using the methods presented in [2]. The result is

$$\Lambda = \begin{bmatrix} \frac{r}{qp^2} & -\frac{1}{p} \\ -\frac{1}{p} & \Lambda_{22} \end{bmatrix}$$

where

$$\Lambda_{22} = \Psi'(r) - E(\Psi'(r+x))$$

The determinant of Λ is found (using [2]) as

$$|\Lambda| = \frac{1}{q^2} \sum_{n=1}^{\infty} \frac{q^n}{(n+1)} \cdot \frac{r!n!}{(r+n)!} \quad (2.4)$$

The two estimating equations g_1 and g_2 , which are exploited herein, are given as

$$g_1(r, p; \vec{x}) = \bar{x} - \frac{rq}{p} = 0 \quad (2.5)$$

$$g_2(r, p; \vec{x}) = \overline{t^x} - \left(\frac{p}{1-qt}\right)^r = 0$$

where

$$\overline{t^x} = \frac{1}{n} \sum_{i=1}^n t^{x_i}$$

is the empirical generating function. (Note: the substitution $t = \exp(-u)$ converts this to La Place transform). Reduction to a single equation is effected by solving $g_1 = 0$ for p in terms of \bar{x} and r ,

$$p = \frac{r}{\bar{x} + r} \quad (2.6)$$

and substituting this into g_2 . A function $f(r)$ is obtained in terms of known variables, namely $\overline{t^x}$, \bar{x} , and t ,

$$f(r) = \left(\frac{r}{r + \bar{x}(1-t)} \right)^r - \bar{t}^{\bar{x}} = 0 \quad (2.7)$$

We can solve equation (2.7) for r by Newton's method, the iterative relationship being

$$r_{i+1} = r_i - \frac{f(r_i)}{f'(r_i)}$$

where

$$f'(r) = \left(\frac{r}{r + \bar{x}(1-t)} \right)^r \cdot \left[\left(\frac{\bar{x}(1-t)}{r + \bar{x}(1-t)} \right) + \log \left(\frac{r}{r + \bar{x}(1-t)} \right) \right]$$

assuming an initial value of r and a suitable t can be found (see table 4.2). After solving for r , p can be found using (2.6).

Efficiency for this r - p pair is given by (see [2,3])

$$\text{Eff} = \frac{|M^{-1}|}{|\Lambda|} \quad (2.8)$$

where

$$|M^{-1}| = \frac{|A|^2}{|C|} \quad (2.9)$$

$|\Lambda|$ is given in (2.4), the matrices A and C in (2.9) are defined as

$$A = E \begin{bmatrix} \frac{\partial g_1}{\partial p} & \frac{\partial g_1}{\partial r} \\ \frac{\partial g_2}{\partial p} & \frac{\partial g_2}{\partial r} \end{bmatrix}$$

$$C = nE \begin{bmatrix} g_1^2 & g_1 g_2 \\ g_1 g_2 & g_2^2 \end{bmatrix}$$

and g_1, g_2 are from (2.5). After some calculation, A and C turn out to be

$$A = \begin{bmatrix} \frac{r}{p^2} & -\frac{q}{p} \\ -\left(\frac{p}{1-qt}\right)^{r-1} \cdot \frac{r(1-t)}{(1-qt)^2} & \left(\frac{p}{1-qt}\right)^r \cdot \log\left(\frac{1-qt}{p}\right) \end{bmatrix} \quad (2.10)$$

$$C = \begin{bmatrix} \frac{rq}{p^2} & -\frac{rqp^{r-1}(1-t)}{(1-qt)^{r+1}} \\ -\frac{rqp^{r-1}(1-t)}{(1-qt)^{r+1}} & \left(\frac{p}{1-qt}\right)^r - \left(\frac{p}{1-qt}\right)^{2r} \end{bmatrix} \quad (2.11)$$

The efficiency of the estimation scheme is found, substituting $|A|$, $|C|$ and $|A|$ into (2.9) and (2.8), as

$$\text{Eff} = \frac{rp^{r+2}}{q} \cdot \frac{\left[\left(\frac{1-qt}{p} \right) \log \left(\frac{1-qt}{p} \right) - \left(\frac{q(1-t)}{p} \right) \right]^2}{\left[\frac{(1-qt)^{2(r+1)}}{(1-qt^2)^r} - p^r (1-qt)^2 - rqp^r (1-t)^2 \right] \left[\sum_{n=1}^{\infty} \frac{q^n}{(n+1)} \cdot \frac{r!n!}{(r+n)!} \right]}$$

(2.12)

Detailed computations are given in Appendix A. The initial value of r is found using the method of moments (\tilde{p}, \tilde{r}) , and the value of t used maximizes the efficiency (2.12) when evaluated at \tilde{p}, \tilde{r} .

III. USAGE AND APPLICATION

In this chapter the procedure to be followed, in order to use the method developed, is given in greater detail and is applied to three sets of data taken from [4] and sixty sets of data generated by computer simulation using various combinations of the parameters.

A. PROCEDURE

The steps of procedure are given below.

1. Using the method of moments, find \tilde{p} and \tilde{r} as starting values. Starting values are given by (3.1) (see [2])

$$\tilde{p} = \bar{x}/s^2 \tag{3.1}$$

$$\tilde{r} = \frac{\bar{x}^2}{s^2 - \bar{x}}$$

where \bar{x} is the sample mean and s^2 is the sample variance.

2. Using \tilde{p} , \tilde{r} found in step 1, find the value of t , which maximizes efficiency function (2.12), and call it t_0 , and also find the efficiency, which is maximum with this pair \tilde{p} , \tilde{r} and t_0 , and call it EFF1. Tables (4.1) and (4.2) have been prepared for this. Single variable search and golden section search were used by the author to find t_0 and EFF1.

3. Using \tilde{r} as a starting value and t_0 , found in step 2, find r^* as a solution of (2.7), and p^* using (2.6). Newton's method was used by the author to find r^* .

4. Step 2 can be repeated with p^* and r^* replacing \tilde{p} and \tilde{r} , and using a new t_0 (update t_0 from Table (4.2) if the current estimated efficiency (Table 4.1) is not sufficiently high). This step involves the recomputations of $\overline{t^x}$ and is seldom needed. The new efficiency using p^* , r^* and updated t_0 is called EFF2.

5. Stop if all the following conditions are met.

$$(1 - \text{EFF1}) \leq e_1, \quad \text{or} \quad |\text{EFF2} - \text{EFF1}| \leq e_2$$

and

$$|p^* - \tilde{p}| \leq e_3,$$

and

$$|r^* - \tilde{r}| \leq e_4.$$

Then p^* and r^* are the estimators of true parameters p and r , and the estimated efficiency is EFF2.

6. If one or more of the conditions in Step 5 are not met, let

$$\tilde{p} = p^*$$

$$\tilde{r} = r^*$$

$$EFF1 = EFF2$$

and go to step 3 and repeat the steps following until the conditions in step 5 are met. e_1, e_2, e_3 and e_4 are the stopping criteria which can be chosen by the user.

B. EXAMPLES

Examples one through three are the applications of the method to the Cricket score data of Reep, Polard and Benjamin [4]. Example four is based on data generated by computer and contains sixty cases.

Example 1: Applying method developed to the Cowdrey data [4] the following information is obtained.

$$\text{Sample mean} = 1.692$$

$$\text{Sample variance} = 4.343$$

$$p^* = 0.325$$

$$r^* = 0.816$$

$$\text{Optimum } t_0 = 0.459$$

$$\text{Efficiency} = 0.996$$

$$\text{Number of Iterations} = 3$$

With the same data maximum likelihood estimators \hat{p}, \hat{r} , and method of moment estimates \tilde{p}, \tilde{r} are given in [2] as

$$\hat{p} = 0.329$$

$$\hat{r} = 0.831$$

$$\tilde{p} = 0.390$$

$$\tilde{r} = 1.080$$

Example 2: Example 1 was repeated for Barrington data [4] and following information is obtained.

$$\text{Sample mean} = 2.095$$

$$\text{Sample variance} = 4.939$$

$$p^* = 0.346$$

$$r^* = 1.111$$

$$\text{Optimum } t_o = 0.538$$

$$\text{Efficiency} = 0.995$$

$$\text{Number of Iterations} = 3$$

$$\hat{p} = 0.345$$

$$\hat{r} = 1.014$$

$$\tilde{p} = 0.424$$

$$\tilde{r} = 1.543$$

Example 3: Like the previous two examples, the following information is obtained by applying the method to the Graveney data [4]:

$$\text{Sample mean} = 1.570$$

$$\text{Sample variance} = 4.474$$

$$p^* = 0.315$$

$$r^* = 0.722$$

Optimum $t_0 = 0.430$

Efficiency = 0.996

Number of iterations = 2

$\hat{p} = 0.317$ $\hat{r} = 0.729$

$\tilde{p} = 0.351$ $\tilde{r} = 0.849$

When the first three examples are examined carefully it is noticed that the method developed in this study is almost as efficient as maximum likelihood method. Estimators are pretty close to maximum likelihood estimators and much better than those found by the method of moments.

Example 4: This example is based on the data generated by computer. Sixty cases, each of which has a different sample size and parameters p and r . Four sample sizes, which are 15, 30, 50 and 100; three r values, 0.5, 2.5 and 5; and five p values, 0.05, 0.1, 0.3, 0.5, 0.8 are used. In each case the data is generated for given sample size (n), p and r . Results, which contain case number (case no.), sample size (n), p , r , method of moments estimates (\tilde{p}, \tilde{r}), optimum u_0 at $\tilde{p}-\tilde{r}$ (Initial u_0), efficiency at the end of the first iteration (Initial eff.), empirical La Place transform estimates (p^*, r^*), optimum u_0 at p^*-r^* (Final u_0), efficiency at p^*-r^* -final u_0 (final eff) and number of iterations (No. of itr.), are tabulated in the following pages. Data generated for the sixty cases are given in

Appendix B. The computer program, written in FORTRAN IV, is also given in Appendix C.

When all the cases are examined, it's seen that estimates are not given for cases 2, 3 and 18. Case 3 provides nothing to work with as all samples are zero, yielding $\bar{X} = 0, s^2 = 0$. The other two cases have $s^2 < \bar{X}$ which signals trouble because the method of moments estimates $\tilde{r} < 0, \tilde{p} > 1$; values which cannot be used to initialize our iteration scheme (3.1) or the maximum likelihood scheme (2.3).

For each of these cases the function $f(r)$ of (2.7) is a non-negative decreasing function of r which tends to $+\infty$ as $r \rightarrow 0$ and is asymptotically zero as $r \rightarrow \infty$. Thus no root exists. The maximum likelihood approach yields a comparable situation.

CASE NO.	<u>r</u>	<u>p</u>	<u>r</u>	<u>\tilde{p}</u>	<u>\tilde{r}</u>	Initial <u>u_o</u>	Initial Eff	<u>p^*</u>	<u>r^*</u>	Final <u>u_o</u>	Final Eff	No. of Itr.
1	15	0.8	5.0	.731	5.081	.2622	.999	.781	6.646	.2162	1.000	3
2	15	0.8	2.5	—	—	—	—	—	—	—	—	—
3	15	0.8	0.5	—	—	—	—	—	—	—	—	—
4	15	0.5	5.0	.468	4.047	.2286	.998	.490	4.425	.2177	.999	3
5	15	0.5	2.5	.539	4.135	.2504	.998	.499	3.524	.2744	.998	4
6	15	0.5	0.5	.459	0.679	1.0334	.997	.540	0.941	.8801	.999	4
7	15	0.3	5.0	.335	5.881	.1187	.998	.324	5.580	.1214	.998	3
8	15	0.3	2.5	.429	4.500	.1919	.998	.429	4.502	.1919	.998	1
9	15	0.3	0.5	.433	1.833	.4550	.991	.319	1.126	.5871	.995	4
10	15	0.1	5.0	.090	4.586	.0452	.988	.074	3.721	.0492	.989	3
11	15	0.1	2.5	.156	3.639	.1106	.989	.135	3.079	.1073	.989	3
12	15	0.1	0.5	.128	.0803	.4937	.983	.111	0.685	.5421	.983	3
13	15	0.05	5.0	.046	4.268	.0265	.991	.052	4.813	.0257	.991	3
14	15	0.05	2.5	.081	4.640	.0415	.988	.067	3.751	.0442	.988	3
15	15	0.05	0.5	.077	0.826	.3529	.964	.050	0.523	.4929	.973	3

CASE NO.	<u>n</u>	<u>p</u>	<u>r</u>	<u>\tilde{p}</u>	<u>\tilde{r}</u>	Initial <u>u_o</u>	Initial Eff.	<u>p^*</u>	<u>r^*</u>	Final <u>u_o</u>	Final Eff.	No. of Itr.
16	30	0.8	5.0	.767	5.587	.2496	1.000	.752	5.151	.2646	1.000	4
17	30	0.8	2.5	.850	5.107	.2929	1.000	.840	4.711	.3114	1.000	4
18	30	0.8	0.5	—	—	—	—	—	—	—	—	—
19	30	0.5	5.0	.449	3.746	.2384	.997	.518	4.938	.2046	.999	5
20	30	0.5	2.5	.592	3.867	.2863	.997	.529	2.995	.3332	.998	4
21	30	0.5	0.5	.559	0.888	.9309	.998	.485	0.659	1.0754	.999	3
22	30	0.3	5.0	.349	6.141	.1176	.998	.337	5.835	.1203	.998	3
23	30	0.3	2.5	.283	2.078	.3019	.994	.311	2.382	.2820	.995	2
24	30	0.3	0.5	.352	0.816	.8083	.996	.343	0.785	.8247	.996	2
25	30	0.1	5.0	.096	4.984	.0446	.994	.103	5.434	.0436	.994	3
26	30	0.1	2.5	.113	2.946	.0963	.986	.106	2.734	.0992	.986	3
27	30	0.1	0.5	.099	0.622	.5681	.983	.092	0.570	.5989	.983	2
28	30	0.05	5.0	.058	6.063	.0218	.995	.059	6.281	.0217	.995	2
29	30	0.05	2.5	.086	4.421	.0465	.986	.068	3.414	.0504	.997	3
30	30	0.05	0.5	.049	0.486	.5324	.975	.048	0.481	.5324	.975	1

CASE NO.	n	p	r	\tilde{p}	\tilde{r}	Initial u_o	Initial Eff	p^*	r^*	Final u_o	Final Eff	No. of Itr.
31	50	0.8	5.0	.632	1.858	.5665	.998	.680	2.300	.4973	.999	3
32	50	0.8	2.5	.700	1.916	.5867	.999	.741	2.346	.5164	1.000	3
33	50	0.8	0.5	.731	0.436	1.5736	1.000	.668	0.322	1.7833	1.000	3
34	50	0.5	5.0	.481	4.461	.2125	.999	.508	4.984	.1998	.999	3
35	50	0.5	2.5	.529	3.052	.3275	.998	.493	2.646	.3551	.998	3
36	50	0.5	0.5	.682	0.900	1.0032	.998	.586	0.594	1.2317	.999	4
37	50	0.3	5.0	.252	4.000	.1384	.996	.261	4.198	.1356	.996	2
38	50	0.3	2.5	.239	2.052	.2704	.992	.249	2.167	.2631	.992	2
39	50	0.3	0.5	.332	0.656	.9307	.996	.290	0.539	1.0188	.996	3
40	50	0.1	5.0	.110	5.487	.0459	.995	.113	5.631	.0455	.995	2
41	50	0.1	2.5	.070	1.694	.1271	.979	.098	2.429	.1067	.984	4
42	50	0.1	0.5	.106	0.555	.6596	.985	.100	0.516	.6878	.986	2
43	50	0.05	5.0	.051	4.997	.0244	.991	.048	4.621	.0249	.991	3
44	50	0.05	2.5	.043	2.149	.0598	.978	.050	2.472	.0562	.978	3
45	50	0.05	0.5	.049	0.518	.4948	.974	.046	0.487	.5183	.974	2

CASE NO.	<u>n</u>	<u>p</u>	<u>r</u>	<u>\tilde{p}</u>	<u>\tilde{r}</u>	Initial <u>u_o</u>	Initial Eff	<u>p^*</u>	<u>r^*</u>	Final <u>u_o</u>	Final Eff	No. of Itr.
46	100	0.8	5.0	.830	5.758	.2584	1.000	.825	5.550	.2658	1.000	3
47	100	0.8	2.5	.832	2.925	.4636	1.000	.828	2.831	.4740	1.000	2
48	100	0.8	0.5	.874	0.626	1.3707	1.000	.866	0.582	1.4202	1.000	2
49	100	0.5	5.0	.463	4.616	.2083	.998	.513	5.401	.1864	.999	4
50	100	0.5	2.5	.525	2.788	.3540	.998	.565	3.279	.3220	.999	3
51	100	0.5	0.5	.500	0.510	1.2802	.999	.507	0.525	1.2642	.999	2
52	100	0.3	5.0	.302	5.115	.1249	.996	.283	4.670	.1298	.997	3
53	100	0.3	2.5	.282	2.533	.2458	.994	.283	2.544	.2452	.994	2
54	100	0.3	0.5	.261	0.384	1.2397	.997	.263	0.388	1.2341	.997	2
55	100	0.1	5.0	.098	5.356	.0423	.994	.101	5.528	.0419	.994	2
56	100	0.1	2.5	.113	2.847	.1000	.987	.112	2.820	.1004	.987	2
57	100	0.1	0.5	.100	0.553	.6442	.984	.091	0.496	.6871	.985	2
58	100	0.05	5.0	.056	5.623	.0233	.993	.056	5.546	.0234	.993	2
59	100	0.05	2.5	.060	3.020	.0523	.984	.059	2.976	.0526	.984	2
60	100	0.05	0.5	.057	0.657	.3947	.970	.050	0.579	.4336	.971	2

IV. RESULT AND CONCLUSION

In this chapter efficiencies for various p-r pairs were given in Table (4.1). The method developed was used to calculate table entries. Three entries were given for each p-r pair. Entry in the middle is efficiency value for that pair at optimum t_0 or u_0 . Entries inside parenthesis are also efficiencies and found as follows:

$$\text{Let } u_0 = -\log t_0$$

If we miss u_0 by ± 50 percent, what would the efficiency be? Efficiency inside the parenthesis above the optimum efficiency answers the question for $u_1 = 0.5 u_0$, and other one below the optimum answers for $u_2 = 1.5 u_0$.

The change $u = -\log t$ (or rather its inverse $t = e^{-u}$) changes the probability generating $\overline{t^x}$ function into the empirical La Place transform. It is seen that a 50 percent error in its argument u results in only a minor degradation of efficiency, whereas a 50 percent error in t would have disastrous effects. In all of the examples, the initial u_0 was well within this 50 percent range, the worse case being 38 percent.

After examining Table (4.1) one can conclude that efficiency is monotonically increasing with increasing p and increasing r for r greater than one and minimum efficiency at optimum t_0 or u_0 is greater than 0.95.

Following Table (4.1) optimum t_0 and u_0 (in parenthesis) were tabulated in Table (4.2). One may observe that t_0 decreases with p and increases with r , whereas u_0 behaves oppositely.

After Table (4.2) efficiency contours were graphed in p - r plane (Figure 4.1), in mean- r plane (Figure 4.2) and in mean- $\exp(r)$ plane (Figure 4.3). Compared with the Figure in [5] Figure (4.3) always gives better efficiency values than all the other alternatives.

TABLE 4.1 ASYMPTOTIC EFFICIENCY OF p^* , r^*

	<u>0.05</u>	<u>0.06</u>	<u>0.07</u>	<u>0.08</u>	<u>0.09</u>	<u>0.1</u>	<u>0.2</u>
0.5	(.961)	(.964)	(.967)	(.969)	(.971)	(.972)	(.982)
	.974	.978	.981	.983	.985	.986	.994
	(.933)	(.935)	(.937)	(.939)	(.941)	(.942)	(.952)
1.0	(.943)	(.947)	(.950)	(.953)	(.955)	(.957)	(.970)
	.963	.967	.970	.973	.976	.978	.989
	(.912)	(.916)	(.919)	(.921)	(.924)	(.926)	(.940)
1.5	(.943)	(.946)	(.948)	(.950)	(.952)	(.954)	(.966)
	.967	.970	.972	.974	.976	.978	.989
	(.912)	(.915)	(.918)	(.920)	(.922)	(.924)	(.939)
2.0	(.947)	(.949)	(.950)	(.952)	(.953)	(.955)	(.965)
	.973	.975	.977	.978	.980	.981	.989
	(.919)	(.921)	(.923)	(.925)	(.927)	(.928)	(.942)
2.5	(.951)	(.952)	(.954)	(.955)	(.956)	(.957)	(.966)
	.979	.980	.981	.982	.983	.984	.991
	(.926)	(.928)	(.929)	(.931)	(.932)	(.934)	(.945)

TABLE 4.1. ASYMPTOTIC EFFICIENCY OF p^* , r^* (Cont'd)

	<u>0.05</u>	<u>0.06</u>	<u>0.07</u>	<u>0.08</u>	<u>0.09</u>	<u>0.1</u>	<u>0.2</u>
3.0	(.955)	(.956)	(.957)	(.958)	(.959)	(.959)	(.967)
	.983	.984	.985	.985	.986	.987	.992
	(.933)	(.934)	(.935)	(.937)	(.938)	(.939)	(.949)
3.5	(.958)	(.959)	(.960)	(.960)	(.961)	(.962)	(.968)
	.986	.987	.987	.988	.989	.989	.993
	(.939)	(.940)	(.941)	(.942)	(.943)	(.932)	(.877)
4.0	(.961)	(.961)	(.962)	(.963)	(.963)	(.964)	(.970)
	.989	.989	.990	.990	.990	.991	.994
	(.944)	(.945)	(.946)	(.947)	(.998)	(.948)	(.956)
4.5	(.963)	(.964)	(.964)	(.965)	(.965)	(.966)	(.971)
	.990	.991	.991	.992	.992	.992	.995
	(.948)	(.949)	(.950)	(.951)	(.951)	(.952)	(.959)
5.	(.965)	(.966)	(.966)	(.967)	(.967)	(.968)	(.973)
	.992	.992	.992	.993	.993	.993	.996
	(.952)	(.953)	(.954)	(.954)	(.955)	(.956)	(.962)

TABLE 4.1. ASYMPTOTIC EFFICIENCY OF p^* , r^* (Cont'd)

	<u>0.05</u>	<u>0.06</u>	<u>0.07</u>	<u>0.08</u>	<u>0.09</u>	<u>0.1</u>	<u>0.2</u>
6.	(.969)	(.969)	(.970)	(.970)	(.970)	(.971)	(.975)
	.994	.994	.994	.995	.995	.995	.997
	(.958)	(.959)	(.959)	(.960)	(.961)	(.961)	(.967)
7.	(.972)	(.972)	(.972)	(.973)	(.973)	(.973)	(.977)
	.995	.996	.996	.996	.996	.996	.997
	(.963)	(.964)	(.964)	(.965)	(.965)	(.966)	(.970)
8.	(.974)	(.974)	(.975)	(.975)	(.975)	(.976)	(.979)
	.996	.996	.997	.997	.997	.997	.998
	(.967)	(.997)	(.968)	(.968)	(.969)	(.969)	(.973)
9.	(.976)	(.976)	(.977)	(.977)	(.977)	(.977)	(.980)
	.997	.997	.997	.997	.997	.997	.998
	(.970)	(.970)	(.971)	(.971)	(.971)	(.972)	(.975)
10.	(.978)	(.978)	(.978)	(.978)	(.979)	(.979)	(.982)
	.998	.998	.998	.998	.998	.998	.998
	(.973)	(.973)	(.973)	(.974)	(.974)	(.974)	(.978)

TABLE 4.1. ASYMPTOTIC EFFICIENCY OF p^* , r^* (Cont'd)

	<u>0.05</u>	<u>0.06</u>	<u>0.07</u>	<u>0.08</u>	<u>0.09</u>	<u>0.1</u>	<u>0.2</u>
11.	(.979)	(.979)	(.980)	(.980)	(.980)	(.980)	(.983)
	.998	.998	.998	.998	.998	.998	.999
	(.975)	(.975)	(.975)	(.976)	(.976)	(.976)	(.976)
12.	(.980)	(.981)	(.981)	(.981)	(.981)	(.982)	(.984)
	.998	.998	.998	.998	.998	.998	.999
	(.978)	(.977)	(.977)	(.977)	(.978)	(.978)	(.981)
13.	(.982)	(.982)	(.983)	(.983)	(.983)	(.984)	(.986)
	.998	.999	.999	.999	.999	.999	.999
	(.978)	(.980)	(.980)	(.980)	(.981)	(.981)	(.983)
14.	(.983)	(.983)	(.983)	(.983)	(.983)	(.984)	(.986)
	.999	.999	.999	.999	.999	.999	.999
	(.980)	(.980)	(.980)	(.980)	(.981)	(.981)	(.983)
15.	(.983)	(.984)	(.984)	(.984)	(.984)	(.984)	(.986)
	.999	.999	.999	.999	.999	.999	.999
	(.981)	(.981)	(.981)	(.982)	(.982)	(.982)	(.984)

TABLE 4.1.1. ASYMPTOTIC EFFICIENCY OF p^* , r^* (Cont'd)

	<u>0.05</u>	<u>0.06</u>	<u>0.07</u>	<u>0.08</u>	<u>0.09</u>	<u>0.1</u>	<u>0.2</u>
16	(.984)	(.984)	(.985)	(.985)	(.985)	(.985)	(.987)
	.999	.999	.999	.999	.999	.999	.999
	(.982)	(.982)	(.982)	(.983)	(.983)	(.983)	(.985)
17	(.985)	(.985)	(.985)	(.986)	(.986)	(.986)	(.988)
	.999	.999	.999	.999	.999	.999	.999
	(.983)	(.983)	(.983)	(.983)	(.984)	(.984)	(.986)
18	(.986)	(.986)	(.986)	(.986)	(.986)	(.987)	(.988)
	.999	.999	.999	.999	.999	.999	.999
	(.984)	(.984)	(.984)	(.984)	(.985)	(.985)	(.987)
19	(.986)	(.986)	(.987)	(.987)	(.987)	(.997)	(.989)
	.999	.999	.999	.999	.999	.999	.999
	(.985)	(.985)	(.985)	(.985)	(.985)	(.985)	(.987)

TABLE 4.1. ASYMPTOTIC EFFICIENCY OF p^* , r^* (Cont'd)

	<u>0.3</u>	<u>0.4</u>	<u>0.5</u>	<u>0.6</u>	<u>0.7</u>	<u>0.8</u>	<u>0.9</u>
	(.987)	(.990)	(.993)	(.995)	(.996)	(.998)	(.999)
0.5	.997	.998	.999	1.000	1.000	1.000	1.000
	(.959)	(.966)	(.972)	(.978)	(.984)	(.990)	(.995)
	(.977)	(.983)	(.987)	(.990)	(.993)	(.996)	(.998)
1.0	.994	.997	.998	.999	1.000	1.000	1.000
	(.951)	(.959)	(.967)	(.974)	(.981)	(.988)	(.994)
	(.973)	(.979)	(.984)	(.988)	(.991)	(.995)	(.997)
1.5	.994	.996	.998	.999	1.000	1.000	1.00
	(.950)	(.959)	(.967)	(.974)	(.981)	(.988)	(.994)
	(.972)	(.978)	(.983)	(.987)	(.990)	(.994)	(.997)
2.0	.994	.996	.998	.999	.999	1.000	1.000
	(.952)	(.960)	(.968)	(.975)	(.982)	(.988)	(.994)
	(.972)	(.978)	(.982)	(.986)	(.990)	(.994)	(.997)
2.5	.994	.997	.998	.999	.999	1.000	1.000
	(.955)	(.963)	(.970)	(.977)	(.983)	(.989)	(.995)

TABLE 4.1. ASYMPTOTIC EFFICIENCY OF p^* , r^* (Cont'd)

	<u>0.3</u>	<u>0.4</u>	<u>0.5</u>	<u>0.6</u>	<u>0.7</u>	<u>0.8</u>	<u>0.9</u>
3.0	(.973)	(.978)	(.982)	(.986)	(.990)	(.994)	(.997)
	.995	.997	.998	.999	1.000	1.000	1.000
	(.958)	(.965)	(.972)	(.978)	(.984)	(.990)	(.995)
3.5	(.974)	(.978)	(.983)	(.987)	(.990)	(.994)	(.997)
	.996	.997	.998	.999	1.000	1.000	1.000
	(.961)	(.967)	(.974)	(.979)	(.985)	(.990)	(.995)
4.0	(.975)	(.979)	(.983)	(.987)	(.990)	(.994)	(.997)
	.998	.998	.999	.999	1.000	1.000	1.000
	(.963)	(.970)	(.975)	(.981)	(.986)	(.991)	(.995)
4.5	(.976)	(.980)	(.984)	(.987)	(.991)	(.994)	(.997)
	.997	.998	.999	.999	1.000	1.000	1.000
	(.966)	(.971)	(.977)	(.982)	(.987)	(.991)	(.996)
5.0	(.977)	(.981)	(.984)	(.988)	(.991)	(.994)	(.997)
	.997	.998	.999	.999	1.000	1.000	1.000
	(.968)	(.973)	(.978)	(.983)	(.987)	(.992)	(.996)

TABLE 4.1. ASYMPTOTIC EFFICIENCY OF p^* , r^* (Cont'd)

	<u>0.3</u>	<u>0.4</u>	<u>0.5</u>	<u>0.6</u>	<u>0.7</u>	<u>0.8</u>	<u>0.9</u>
	(.979)	(.982)	(.985)	(.989)	(.992)	(.994)	(.997)
6.0	.998	.998	.999	.999	1.000	1.000	1.000
	(.972)	(.976)	(.981)	(.985)	(.989)	(.993)	(.996)
	(.980)	(.983)	(.986)	(.989)	(.992)	(.995)	(.997)
7.0	.998	.999	.999	1.000	1.000	1.000	1.000
	(.974)	(.979)	(.982)	(.986)	(.990)	(.993)	(.997)
	(.982)	(.985)	(.987)	(.990)	(.993)	(.995)	(.998)
8.0	.998	.000	.000	1.000	1.000	1.000	1.000
	(.977)	(.981)	(.984)	(.987)	(.991)	(.994)	(.997)
	(.983)	(.986)	(.988)	(.991)	(.993)	(.995)	(.998)
9.0	.999	.999	.999	1.000	1.000	1.000	1.000
	(.979)	(.982)	(.985)	(.989)	(.992)	(.994)	(.997)
	(.984)	(.987)	(.989)	(.991)	(.993)	(.996)	(.998)
10.0	.999	.999	1.000	1.000	1.000	1.000	1.000
	(.981)	(.984)	(.987)	(.989)	(.992)	(.995)	(.997)

TABLE 4.1. ASYMPTOTIC EFFICIENCY OF p^* , r^* (Cont'd)

	<u>0.3</u>	<u>0.4</u>	<u>0.5</u>	<u>0.6</u>	<u>0.7</u>	<u>0.8</u>	<u>0.9</u>
	(.985)	(.987)	(.990)	(.992)	(.994)	(.996)	(.998)
11.0	.999	.999	1.000	1.000	1.000	1.000	1.000
	(.982)	(.985)	(.988)	(.990)	(.993)	(.995)	(.998)
	(.986)	(.988)	(.990)	(.992)	(.994)	(.996)	(.998)
12.0	.999	.999	1.000	1.000	1.000	1.000	1.000
	(.983)	(.986)	(.988)	(.991)	(.993)	(.996)	(.998)
	(.987)	(.989)	(.991)	(.993)	(.995)	(.996)	(.998)
13.0	.999	.999	1.000	1.000	1.000	1.000	1.000
	(.984)	(.987)	(.989)	(.991)	(.994)	(.996)	(.998)
	(.987)	(.989)	(.991)	(.993)	(.995)	(.997)	(.998)
14.0	.999	1.000	1.000	1.000	1.000	1.000	1.000
	(.985)	(.988)	(.990)	(.992)	(.994)	(.996)	(.998)
	(.988)	(.990)	(.992)	(.993)	(.995)	(.997)	(.998)
15.0	.999	1.000	1.000	1.000	1.000	1.000	1.000
	(.986)	(.988)	(.990)	(.992)	(.994)	(.996)	(.998)

TABLE 4.1. ASYMPTOTIC EFFICIENCY OF p^* , r^* (Cont'd)

	<u>0.3</u>	<u>0.4</u>	<u>0.5</u>	<u>0.6</u>	<u>0.7</u>	<u>0.8</u>	<u>0.9</u>
	(.989)	(.990)	(.992)	(.994)	(.995)	(.997)	(.998)
16.0	.999	1.000	1.000	1.000	1.000	1.000	1.000
	(.987)	(.989)	(.991)	(.993)	(.995)	(.996)	(.998)
	(.989)	(.991)	(.992)	(.994)	(.995)	(.997)	(.999)
17.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(.988)	(.990)	(.991)	(.993)	(.995)	(.997)	(.998)
	(.990)	(.991)	(.993)	(.994)	(.996)	(.997)	(.999)
18.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(.988)	(.990)	(.992)	(.993)	(.995)	(.997)	(.998)
	(.990)	(.991)	(.993)	(.994)	(.996)	(.997)	(.999)
19.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(.989)	(.991)	(.992)	(.994)	(.995)	(.997)	(.998)

TABLE 4.2 OPTIMUM t_o AND u_o FOR $p^* = r^*$

	0.05	0.06	0.07	0.08	0.09	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.5	.594	.567	.544	.524	.507	.491	.391	.337	.300	.274	.253	.236	.222	.210
	(.520)	(.567)	(.608)	(.646)	(.680)	(.711)	(.938)	(1.088)	(1.202)	(1.295)	(1.374)	(1.443)	(1.504)	(1.559)
1.0	.817	.794	.772	.753	.735	.719	.603	.532	.482	.444	.413	.388	.367	.349
	(.202)	(.231)	(.258)	(.284)	(.308)	(.330)	(.506)	(.631)	(.730)	(.813)	(.883)	(.946)	(1.002)	(1.052)
1.5	.894	.877	.861	.847	.833	.820	.717	.648	.596	.554	.521	.492	.468	.447
	(.112)	(.131)	(.149)	(.166)	(.183)	(.199)	(.332)	(.434)	(.518)	(.590)	(.653)	(.708)	(.759)	(.805)
2.0	.928	.915	.903	.892	.881	.871	.785	.721	.671	.630	.596	.567	.542	.520
	(.075)	(.089)	(.102)	(.114)	(.127)	(.139)	(.243)	(.327)	(.399)	(.461)	(.517)	(.567)	(.612)	(.654)
2.5	.946	.936	.927	.918	.909	.900	.827	.770	.724	.685	.652	.624	.598	.576
	(.056)	(.066)	(.076)	(.086)	(.096)	(.105)	(.190)	(.261)	(.323)	(.378)	(.427)	(.472)	(.514)	(.552)
3.0	.957	.949	.941	.934	.926	.919	.856	.805	.763	.727	.695	.667	.642	.620
	(.044)	(.053)	(.061)	(.069)	(.077)	(.084)	(.155)	(.216)	(.271)	(.319)	(.364)	(.405)	(.443)	(.478)
3.5	.964	.958	.951	.945	.938	.932	.877	.832	.792	.758	.728	.702	.678	.656
	(.036)	(.043)	(.050)	(.057)	(.064)	(.070)	(.131)	(.184)	(.233)	(.276)	(.317)	(.354)	(.389)	(.421)
4.0	.969	.964	.958	.953	.947	.942	.893	.852	.816	.784	.756	.730	.707	.686
	(.031)	(.037)	(.043)	(.049)	(.054)	(.060)	(.113)	(.161)	(.204)	(.244)	(.280)	(.315)	(.347)	(.377)

TABLE 4.2 OPTIMUM t_o AND u_o FOR $p^* \sim r^*$ (Cont'd)

	<u>0.05</u>	<u>0.06</u>	<u>0.07</u>	<u>0.08</u>	<u>0.09</u>	<u>0.1</u>	<u>0.2</u>	<u>0.3</u>	<u>0.4</u>	<u>0.5</u>	<u>0.6</u>	<u>0.7</u>	<u>0.8</u>	<u>0.9</u>
4.5	.973 (.027)	.968 (.032)	.963 (.037)	.959 (.042)	.954 (.047)	.949 (.052)	.905 (.099)	.868 (.142)	.834 (.181)	.804 (.218)	.778 (.251)	.753 (.283)	.731 (.313)	.711 (.341)
5.0	.976 (.024)	.972 (.028)	.968 (.033)	.963 (.037)	.959 (.042)	.955 (.046)	.915 (.089)	.880 (.127)	.849 (.163)	.822 (.197)	.796 (.228)	.773 (.257)	.752 (.285)	.732 (.311)
6.0	.981 (.019)	.977 (.023)	.974 (.027)	.970 (.030)	.966 (.034)	.963 (.038)	.930 (.073)	.900 (.105)	.873 (.136)	.848 (.165)	.826 (.192)	.805 (.217)	.785 (.242)	.767 (.265)
7.0	.984 (.016)	.981 (.019)	.978 (.023)	.975 (.026)	.972 (.029)	.969 (.032)	.940 (.062)	.914 (.090)	.890 (.116)	.868 (.141)	.848 (.165)	.828 (.188)	.810 (.210)	.794 (.231)
8.0	.986 (.014)	.983 (.017)	.981 (.019)	.978 (.022)	.975 (.025)	.973 (.028)	.948 (.054)	.925 (.078)	.903 (.102)	.883 (.124)	.865 (.146)	.847 (.166)	.830 (.186)	.815 (.205)
9.0	.988 (.012)	.985 (.015)	.983 (.017)	.981 (.019)	.978 (.022)	.976 (.024)	.954 (.047)	.933 (.069)	.914 (.090)	.895 (.110)	.878 (.130)	.862 (.148)	.847 (.166)	.832 (.184)
10.0	.989 (.011)	.987 (.013)	.985 (.015)	.983 (.017)	.981 (.019)	.979 (.022)	.959 (.042)	.940 (.062)	.922 (.081)	.905 (.100)	.889 (.117)	.874 (.134)	.860 (.151)	.846 (.167)
11.0	.990 (.010)	.988 (.012)	.986 (.014)	.984 (.016)	.983 (.018)	.981 (.020)	.962 (.038)	.945 (.056)	.929 (.074)	.913 (.091)	.899 (.107)	.885 (.123)	.871 (.138)	.858 (.153)

TABLE 4.2 OPTIMUM t_o AND u_o FOR $p^* - r^*$ (Cont'd)

	<u>0.05</u>	<u>0.06</u>	<u>0.07</u>	<u>0.08</u>	<u>0.09</u>	<u>0.1</u>	<u>0.2</u>	<u>0.3</u>	<u>0.4</u>	<u>0.5</u>	<u>0.6</u>	<u>0.7</u>	<u>0.8</u>	<u>0.9</u>
12.0	.991 (.009)	.989 (.011)	.988 (.013)	.986 (.014)	.084 (.016)	.082 (.018)	.966 (.035)	.950 (.051)	.935 (.068)	.920 (.083)	.907 (.098)	.893 (.113)	.881 (.127)	.869 (.141)
13.0	.992 (.008)	.990 (.010)	.989 (.011)	.987 (.013)	.985 (.015)	.984 (.016)	.968 (.032)	.954 (.047)	.940 (.062)	.926 (.077)	.913 (.091)	.901 (.104)	.889 (.118)	.878 (.130)
14.0	.992 (.008)	.991 (.009)	.989 (.011)	.988 (.012)	.986 (.014)	.985 (.015)	.971 (.030)	.957 (.044)	.944 (.058)	.931 (.071)	.919 (.084)	.907 (.097)	.896 (.110)	.885 (.112)
15.0	.993 (.007)	.992 (.008)	.990 (.010)	.989 (.011)	.987 (.013)	.986 (.014)	.973 (.028)	.960 (.041)	.948 (.054)	.936 (.067)	.924 (.079)	.913 (.091)	.903 (.103)	.892 (.114)
16.0	.993 (.007)	.992 (.008)	.991 (.009)	.990 (.011)	.988 (.012)	.987 (.013)	.974 (.026)	.962 (.038)	.951 (.051)	.940 (.062)	.929 (.074)	.918 (.085)	.908 (.096)	.898 (.107)
17.0	.994 (.006)	.993 (.007)	.991 (.009)	.990 (.011)	.989 (.012)	.988 (.024)	.976 (.036)	.965 (.048)	.954 (.059)	.943 (.070)	.933 (.080)	.923 (.091)	.913 (.091)	.904 (.101)
18.0	.994 (.006)	.993 (.007)	.992 (.008)	.991 (.009)	.990 (.010)	.988 (.012)	.977 (.023)	.967 (.034)	.956 (.045)	.946 (.055)	.936 (.066)	.927 (.076)	.918 (.086)	.909 (.096)
19.0	.994 (.006)	.993 (.007)	.992 (.008)	.991 (.009)	.990 (.010)	.989 (.011)	.979 (.022)	.968 (.032)	.958 (.042)	.949 (.053)	.939 (.062)	.930 (.072)	.922 (.082)	.913 (.091)

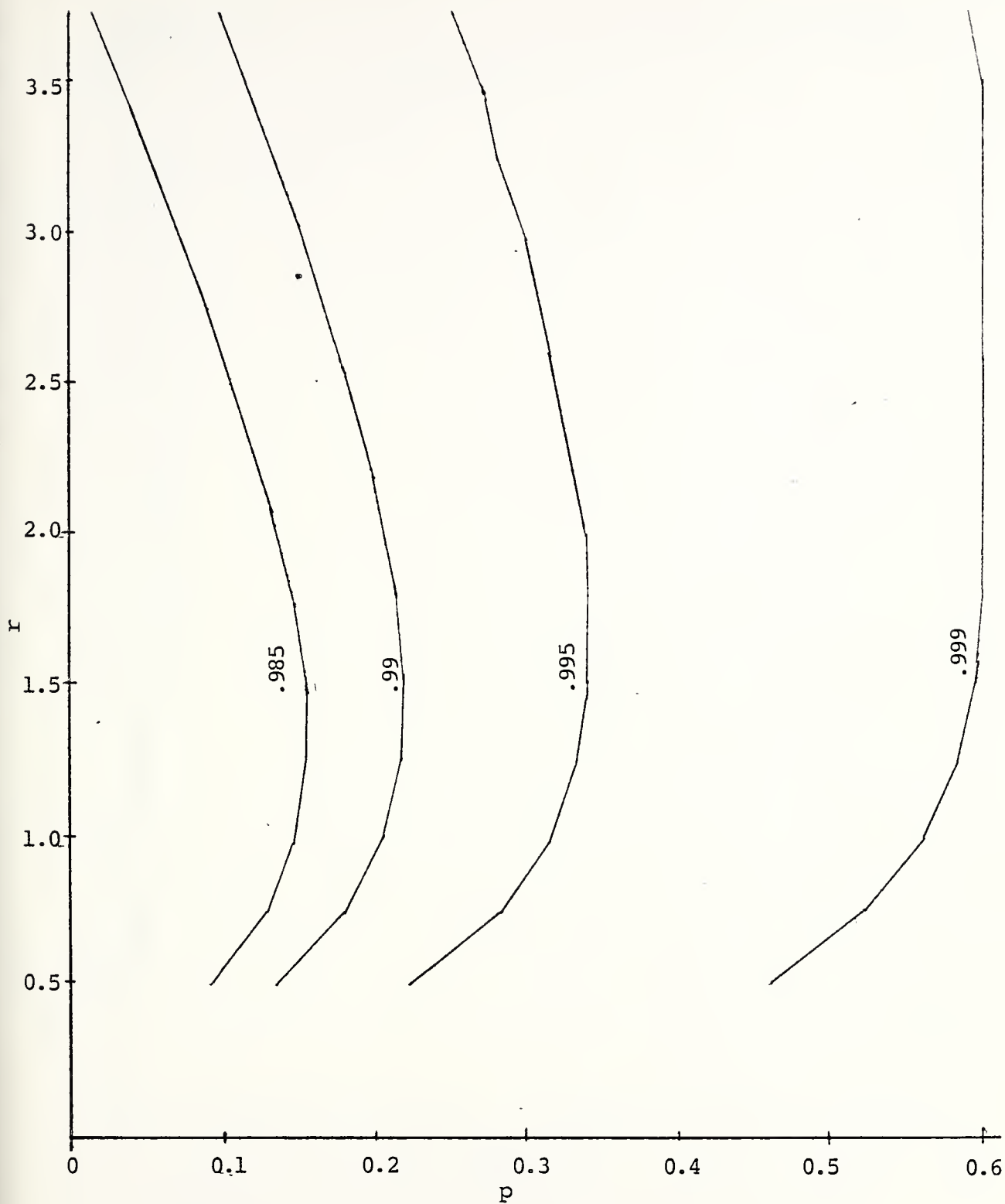


Figure 4.1 Efficiency contours in p - r plane

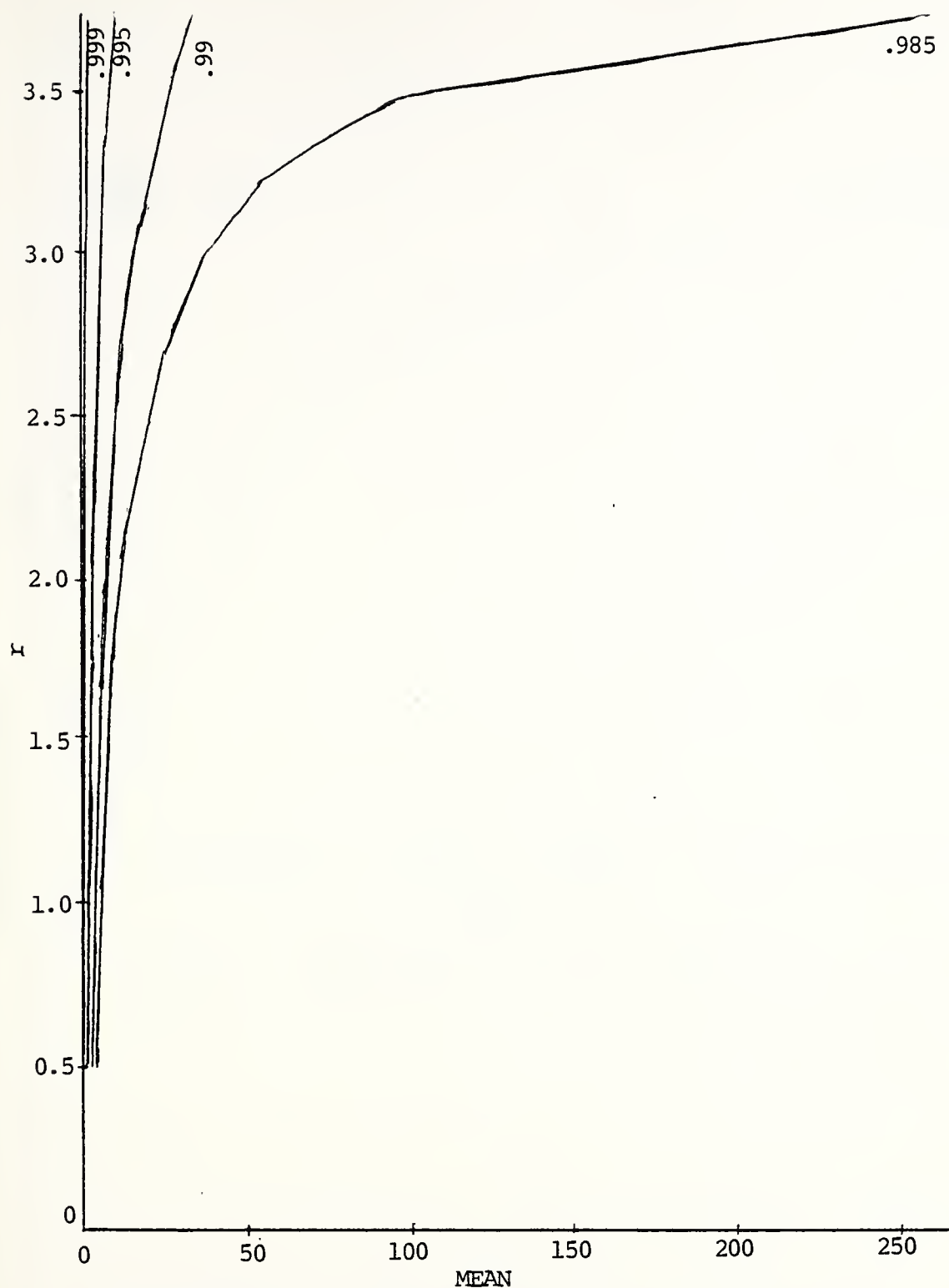


Figure 4.2 Efficiency contours in MEAN-r plane

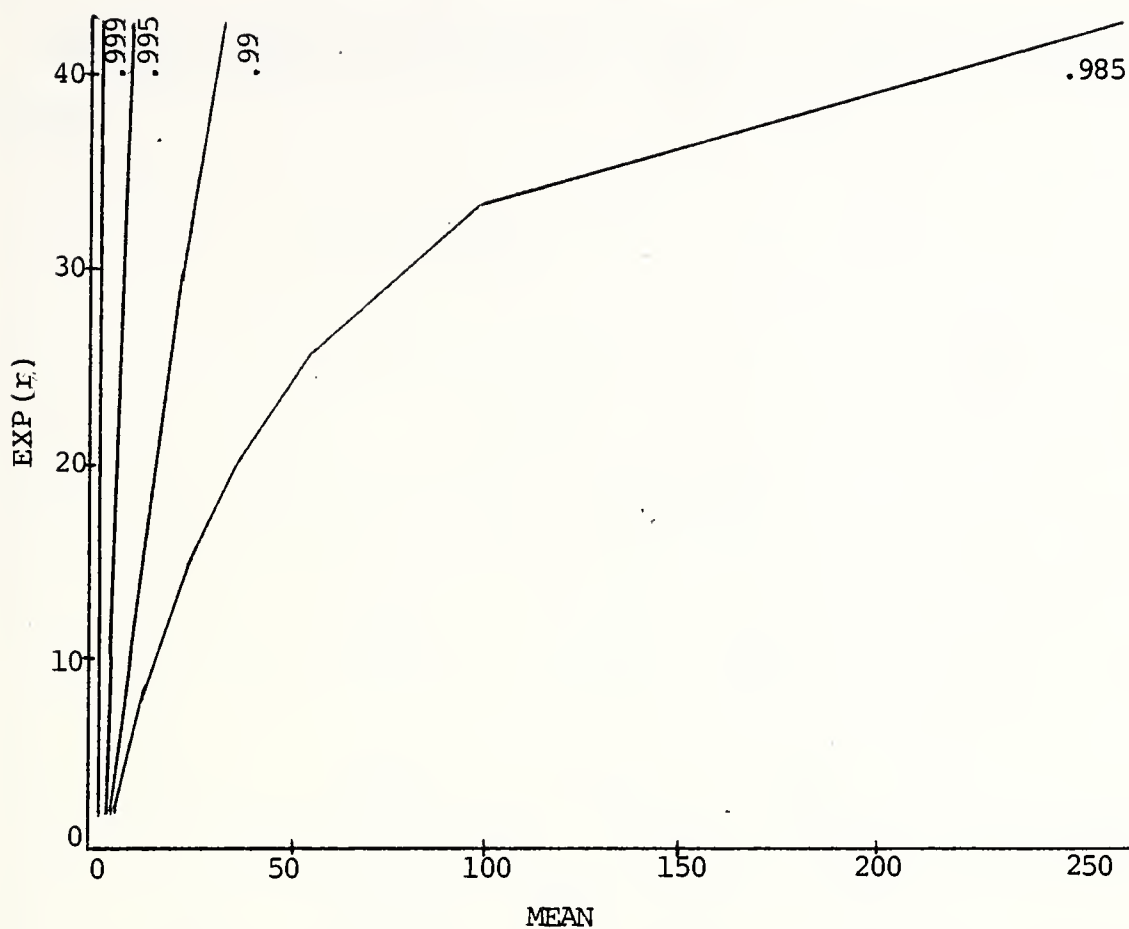


Figure 4.3 Efficiency contours in MEAN-EXP(r) plane

APPENDIX A
DETAILED COMPUTATIONS

The two estimating equations g_1 and g_2 given in chapter II are

$$g_1 = \bar{x} - \frac{rq}{p} = 0$$

$$g_2 = \overline{t^x} - \left(\frac{p}{1-qt}\right)^r = 0$$

Matrix A's entries were given as follows:

$$A_{11} = E\left(\frac{\partial g_1}{\partial p}\right) = E\left(\frac{r}{p^2}\right) = \frac{r}{p^2}$$

$$A_{12} = E\left(\frac{\partial g_1}{\partial r}\right) = E\left(-\frac{q}{p}\right) = -\frac{q}{p}$$

$$\begin{aligned} A_{21} &= E\left(\frac{\partial g_2}{\partial p}\right) = E\left[-\left(\frac{p}{1-qt}\right)^{r-1} \cdot \frac{r(1-t)}{(1-qt)^2}\right] \\ &= -\left(\frac{p}{1-qt}\right)^{r-1} \cdot \frac{r(1-t)}{(1-qt)^2} \end{aligned}$$

$$\begin{aligned} A_{22} &= E\left(\frac{\partial g_2}{\partial r}\right) = E\left[\ln\left(\frac{1-qt}{p}\right) \left(\frac{p}{1-qt}\right)^r\right] \\ &= \ln\left(\frac{1-qt}{p}\right) \left(\frac{p}{1-qt}\right)^r \end{aligned}$$

Matrix C's entries were given as follows:

$$C_{11} = n E(g_1^2) = n E[(\bar{x} - \frac{rq}{p})^2]$$

$$= n \text{Var}(\bar{x}) = n \text{Var}(\frac{1}{n} \sum_{i=1}^n x_i)$$

$$= n \cdot \frac{1}{n^2} \text{Var}(\sum_{i=1}^n x_i) = \text{Var}(x)$$

$$= \frac{rq}{p^2}$$

$$C_{12} = n E[g_1 \cdot g_2]$$

$$= n E[(\bar{x} - \frac{rq}{p})(\bar{t^x} - (\frac{p}{1-qt})^r)]$$

$$= n \text{Cov}(\bar{x}, \bar{t^x}) = n \text{Cov}(\frac{1}{n} \sum_{i=1}^n x_i, \bar{t^x})$$

$$= \text{Cov}(x_i, \bar{t^x})$$

$$= \sum_{i=1}^n \text{Cov}(x_i, \bar{t^x}) = \sum_{i=1}^n \text{Cov}(x_i, \frac{1}{n} \sum_{j=1}^n t^{x_j})$$

$$= \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \text{Cov}(x_i, t^{x_j})$$

Since $\text{Cov}(x_i, t^{x_j}) = 0$ for $i \neq j$ because of independence, it follows that

$$\begin{aligned} C_{12} &= C_{21} = \frac{1}{n} \sum_{i=1}^n \text{Cov}(x_i, t^{x_i}) \\ &= \text{Cov}(x, t^x) = E(x \cdot t^x) - E(x)E(t^x) \end{aligned}$$

One can calculate $E(x \cdot t^x)$ using the following relation.

$$G(t) = E(t^x) \quad G'(t) = E(xt^{x-1})$$

Then

$$E(xt^x) = tG'(t)$$

where

$$G(t) = \left(\frac{p}{1-qt}\right)^r$$

taking the derivative and multiplying by t , $E(xt^x)$ is found as

$$E(xt^x) = \frac{trqp^r}{(1-qt)^{r+1}}$$

Then it yields

$$C_{12} = C_{21} = -\frac{rqp^{r-1}(1-t)}{(1-qt)^{r+1}}$$

$$C_{22} = n E(g_2^2) = E[(\overline{t^x} - (\frac{p}{1-qt})^r)^2]$$

$$= n \text{Var}(\overline{t^x}) = n \text{Var}(\frac{1}{n} \sum_{i=1}^n t^{x_i})$$

$$= \frac{1}{n} \text{Var}(\sum_{i=1}^n t^{x_i})$$

$$= \text{Var}(t^x) = E(t^{2x}) - [E(t^x)]^2$$

$$= G(t^2) - [G(t)]^2$$

$$= (\frac{p}{1-qt^2})^r - (\frac{p}{1-qt})^{2r}$$

As the reader will recall, $|\Lambda|$ was given by (2.4) and can be written as follows.

$$|\Lambda| = \frac{1}{p^2} \left[\sum_{n=1}^m \frac{q^n}{n+1} \cdot \frac{r!n!}{(n+r)!} + \sum_{n=m+1}^{\infty} \frac{q^n}{n+1} \frac{r!n!}{(n+r)!} \right]$$

Call the summation from $n = m+1$ to ∞ the remainder. The remainder can be adjusted such that it will be smaller than any specified small value (ϵ).

$$\sum_{n=m+1}^{\infty} \frac{q^n}{n+1} \frac{r!n!}{(n+r)!} \leq \sum_{n=m+1}^{\infty} \frac{q^n}{n+1}$$

Because $\frac{r!n!}{(n+r)!} \leq 1$ (reciprocal of binomial coefficient),

$$\sum_{n=m+1}^{\infty} \frac{q^n}{n+1} \leq \frac{q^{m+1}}{m+2} \sum_{n=0}^{\infty} q^n = \frac{q^{m+1}}{(m+2)p} \leq \epsilon$$

A value of m can be found such that

$$\frac{q^{m+1}}{(m+2)p} \leq \epsilon$$

holds. Then this m guarantees the remainder will be smaller than ϵ , and the efficiency equation of (2.12) becomes

$$\text{Eff} = \frac{rp^{r+2}}{q} \cdot \frac{\left[\frac{1-qt}{p} \ln\left(\frac{1-qt}{p}\right) - \frac{q(1-t)}{p} \right]^2}{\left[\frac{(1-qt)^{2(r+1)}}{(1+qt)^2} - p^r (1-qt)^2 - rqp^r (1-t)^2 \right] \left(\sum_{n=1}^m \frac{q^n}{n+1} \frac{r!n!}{(r+n)!} \right)}$$

APPENDIX B

SAMPLES GENERATED BY COMPUTER

Samples generated by computer for sixty cases used in chapter three, example four, are given in the following pages case by case.

See subroutine GNRNB of computer program-3 of Appendix C for the simulation procedure, which was taken from [6].

<u>VAL.</u>	<u>FRQ.</u>	<u>VAL.</u>	<u>FRQ.</u>	<u>VAL.</u>	<u>FRQ.</u>	<u>VAL.</u>	<u>FRQ.</u>
<u>CASE 1</u>		<u>CASE 7</u>		<u>CASE 10</u> (Cont'd)		<u>CASE 13</u> (Cont'd)	
0	2	3	2			68	1
1	7	5	1	63	1	70	1
2	1	8	1	64	1	88	1
3	3	9	1	71	1	120	1
5	2	10	2	73	1	131	1
		11	1	74	1	139	1
		12	1	76	1	153	1
		13	1			176	1
		14	1	<u>CASE 11</u>			
		15	1				
		19	1	2	1	<u>CASE 14</u>	
		21	1	7	1		
		22	1	8	1	12	1
				9	1	15	1
		<u>CASE 8</u>		11	1	24	1
		1	1	17	1	30	1
		2	2	18	1	40	1
		3	3	19	1	42	1
		5	1	22	2	50	1
		6	1	25	1	52	1
		7	1	27	1	59	1
		8	3	30	1	62	1
		9	1	37	1	70	1
		11	1	41	1	71	1
		14	1			78	1
				<u>CASE 12</u>		85	1
						96	1
		<u>CASE 9</u>		0	4		
		0	5	1	2	<u>CASE 15</u>	
		1	2	4	1		
		2	2	5	1	0	3
		4	2	6	3	1	2
		5	2	7	1	2	1
		6	2	10	1	3	1
				11	1	5	1
		<u>CASE 10</u>		25	1	9	2
		11	1			15	1
		14	1	<u>CASE 13</u>		18	1
		21	1			22	1
		30	1	33	1	30	1
		33	2	49	1	34	1
		39	1	52	1		
		40	1	59	1	<u>CASE 16</u>	
		57	1	60	1		
				63	1	0	8
				64	1	1	7
						2	6
						3	6
						4	1
						5	2
<u>CASE 2</u>							
0	8						
1	5						
2	2						
<u>CASE 3</u>							
0	15						
<u>CASE 4</u>							
0	1						
1	1						
2	1						
3	4						
4	1						
5	3						
7	2						
9	1						
12	1						
<u>CASE 5</u>							
0	2						
1	1						
2	4						
3	1						
4	2						
5	1						
7	4						
<u>CASE 6</u>							
0	8						
1	5						
2	1						
5	1						

VAL.	FRQ.	VAL.	FRQ.	VAL.	FRQ.	VAL.	FRQ.
<u>CASE 17</u>		<u>CASE 22</u>		<u>CASE 25</u>		<u>CASE 27</u>	
0	14	1	1	14	1	0	8
1	8	3	2	23	1	1	4
2	5	6	1	28	3	2	1
3	3	7	3	29	1	3	3
		8	4	30	1	4	3
	<u>CASE 18</u>	9	2	31	1	5	2
		11	5	32	1	7	1
0	27	12	1	33	4	8	1
1	3	13	1	34	1	9	2
	<u>CASE 19</u>	14	2	36	1	11	1
		15	1	37	2	15	1
		16	1	47	1	18	1
		17	1	51	1	26	1
1	2	19	2	53	1	29	1
2	6	20	2	58	1		
3	7	26	1	60	1		<u>CASE 28</u>
4	3			67	1		
5	4	<u>CASE 23</u>		68	1		
6	3			71	1	42	1
7	1			73	1	46	1
10	1	0	3	77	1	52	1
11	2	1	2	81	1	55	1
14	1	2	2	88	1	57	1
		3	3	102	1	58	1
	<u>CASE 20</u>	4	4			59	1
		5	5		<u>CASE 26</u>	60	1
0	5	6	4			65	1
1	8	7	1	0	1	72	1
2	2	8	2	5	1	73	1
3	2	10	2	6	1	79	1
4	7	15	1	8	1	81	1
5	4	20	1	9	1	91	1
7	2			10	1	92	1
	<u>CASE 21</u>	<u>CASE 24</u>		12	1	98	1
				15	3	100	1
0	19	0	12	16	2	106	1
1	5	1	10	17	1	112	1
2	3	2	2	20	2	118	1
3	2	3	1	21	1	120	1
4	1	4	1	23	1	121	1
		5	2	24	1	124	1
		7	2	27	2	129	1
				29	1	132	1
				31	1	141	1
				32	1	148	1
				33	1	167	1
				35	1	185	1
				47	1	197	1
				49	1		
				53	1		
				61	1		

<u>VAL.</u>	<u>FRQ.</u>	<u>VAL.</u>	<u>FRQ.</u>	<u>VAL.</u>	<u>FRQ.</u>	<u>VAL.</u>	<u>FRQ.</u>
<u>CASE 29</u>		<u>CASE 31</u>		<u>CASE 36</u>		<u>CASE 39</u>	
7	1	0	21	0	37	0	27
12	1	1	13	1	6	1	5
15	1	2	11	2	6	2	8
18	1	3	4	3	1	3	4
21	3	7	1			4	3
22	1			<u>CASE 37</u>		5	1
26	1	<u>CASE 32</u>				6	1
30	1					10	1
33	1			2	1		
40	1	0	24	3	3	<u>CASE 40</u>	
49	1	1	17	4	3		
52	1	2	6	5	2	8	1
54	4	3	1	6	2	16	1
55	2	4	1	7	2	17	1
57	1	5	1	8	5	22	1
58	1			9	3	26	3
65	1	<u>CASE 33</u>		10	3	27	4
66	1			11	2	28	1
68	1	0	44	12	6	30	1
71	2	1	4	13	1	31	1
74	1	2	2	14	2	32	3
86	1			15	4	33	3
95	1	<u>CASE 34</u>		17	4	35	2
				20	1	36	2
<u>CASE 30</u>		1	8	22	2	37	1
0	6	2	6	23	1	40	2
1	5	3	3	25	1	41	1
2	2	4	7	29	1	44	2
3	2	5	10	33	1	45	1
4	1	6	4	<u>CASE 38</u>		47	1
5	2	7	2			48	1
6	1	8	4	0	1	49	1
8	1	9	4	1	7	54	2
9	1	11	1	2	4	55	1
10	1	17	1	3	3	58	1
12	2			4	5	60	1
15	1	<u>CASE 35</u>		5	6	61	1
21	1			6	4	62	1
25	1	0	9	7	6	68	1
38	1	1	9	8	2	70	1
47	1	2	8	10	3	71	2
52	1	3	8	11	2	74	1
		4	5	14	1	75	1
		5	6	15	2	87	1
		6	1	16	2	91	1
		7	2	17	1	93	1
		8	1	25	1		
		9	1				

<u>VAL.</u>	<u>FRQ.</u>	<u>VAL.</u>	<u>FRQ.</u>	<u>VAL.</u>	<u>FRQ.</u>	<u>VAL.</u>	<u>FRQ.</u>
<u>CASE 41</u>		<u>CASE 43</u>		<u>CASE 44</u>		<u>CASE 45</u> (Cont'd)	
3	1	17	1	7	1		
4	2	21	1	8	2	8	3
8	3	33	1	12	1	9	1
9	3	40	1	13	1	10	4
10	3	43	1	15	1	11	3
11	1	45	1	21	3	12	1
12	1	50	1	22	1	13	1
13	1	51	2	24	2	14	1
14	4	52	1	25	2	23	2
15	4	55	1	27	1	26	1
16	1	57	1	29	3	31	1
17	2	61	1	30	2	34	1
18	3	63	2	32	1	38	1
19	1	64	1	35	2	59	1
20	2	68	1	38	1	64	1
21	1	69	1	40	1		
22	1	71	1	41	1	<u>CASE 46</u>	
23	1	76	1	42	1		
24	1	78	1	43	1	0	35
26	1	80	1	44	1	1	32
27	3	82	1	45	1	2	20
29	1	83	1	46	1	3	7
33	1	87	1	47	1	4	5
35	1	90	1	48	1	5	1
41	1	93	1	49	1		
47	1	94	1	51	2	<u>CASE 47</u>	
50	2	95	1	53	1		
74	1	98	1	55	1		
76	1	100	1	56	1	0	59
82	1	101	1	70	1	1	27
		102	1	73	1	2	11
<u>CASE 42</u>		104	1	77	1	3	-2
		109	1	82	1	4	1
0	16	114	2	85	1		
1	4	121	1	86	1	<u>CASE 48</u>	
2	9	125	1	105	1		
3	2	127	1	111	1	0	92
4	1	129	1	130	1	1	7
5	3	135	1	140	2	2	1
6	2	140	1				
7	1	149	1	<u>CASE 45</u>			
8	3	158	1				
10	2	163	1	0	12		
12	3	167	1	1	4		
16	2	171	1	2	3		
20	1	184	1	3	5		
35	1	185	1	4	2		
				6	1		
				7	1		

VAL.	FRQ.	VAL.	FRQ.	VAL.	FRQ.	VAL.	FRQ.
<u>CASE 49</u>		<u>CASE 52</u>		<u>CASE 54</u>		<u>CASE 55</u>	
		(Cont'd)				(Cont'd)	
0	1			0	60		
1	8	10	7	1	15	56	1
2	12	11	2	2	10	59	5
3	13	12	5	3	4	61	1
4	13	13	5	4	6	62	2
5	17	14	4	5	2	64	1
6	11	15	5	6	1	65	1
7	9	16	5	8	1	68	3
8	4	17	4	14	1	72	4
9	2	18	1			74	1
10	2	19	1	<u>CASE 55</u>		76	1
11	5	20	9			77	1
15	2	21	1	13	1	78	1
20	1	22	2	15	3	81	2
		24	2	18	1	88	1
<u>CASE 50</u>		26	1	19	1	90	1
0	15	27	1	22	1	91	1
1	21	29	1	23	1	97	2
2	22	<u>CASE 53</u>		24	2	101	1
3	17			25	4	112	1
4	12			26	2	118	1
5	5	0	5	28	1	<u>CASE 56</u>	
6	2	1	9	29	1		
7	2	2	3	30	1		
8	3	3	10	31	1	1	1
13	1	4	16	32	2	4	2
		5	9	33	2	5	7
<u>CASE 51</u>		6	7	34	1	6	1
0	70	7	7	35	3	7	2
1	18	8	7	36	6	8	3
2	8	9	3	37	1	9	2
3	1	10	7	38	4	10	4
4	2	11	5	39	3	11	2
6	1	12	2	40	2	12	3
		13	2	41	1	13	2
<u>CASE 52</u>		14	2	42	2	14	4
0	1	15	2	43	1	15	2
2	3	17	1	44	2	16	4
3	3	19	1	45	1	17	3
4	4	23	1	46	3	18	2
5	4	24	1	47	2	19	2
6	3			49	1	20	4
7	7			50	2	21	1
8	12			51	1	22	3
9	7			52	1	23	4
				53	3	24	5
				54	2	25	2
				55	2	26	1

<u>VAL.</u>	<u>FRQ.</u>	<u>VAL.</u>	<u>FRQ.</u>	<u>VAL.</u>	<u>FRQ.</u>	<u>VAL.</u>	<u>FRQ.</u>
<u>CASE 56</u>		<u>CASE 58</u>		<u>CASE 58</u>		<u>CASE 59</u>	
(Cont'd)				(Cont'd)		(Cont'd)	
		21	1				
27	4	33	2	116	1	30	3
28	2	35	1	118	2	32	3
29	2	37	1	119	1	34	2
30	2	43	1	121	1	35	5
31	3	44	2	122	1	36	1
33	3	45	4	124	1	37	2
34	3	46	4	125	1	38	1
35	1	50	1	127	2	39	1
36	1	53	1	131	1	40	2
37	1	54	1	135	1	41	2
38	1	55	2	137	1	42	2
39	2	57	1	140	1	44	1
40	2	60	1	141	1	45	1
46	1	64	1	144	1	47	1
48	1	65	1	147	1	49	2
50	1	68	1	152	1	51	1
53	1	69	3	153	1	52	1
54	1	71	1	161	1	53	1
72	1	72	2	180	1	54	1
76	1	73	1	185	2	56	1
		74	2	190	1	57	2
<u>CASE 57</u>		76	2	192	1	58	1
		77	2	193	1	61	1
		78	1	197	1	63	1
0	32	79	1			64	1
1	10	80	1	<u>CASE 59</u>		65	1
2	10	81	1			66	1
3	6	82	1			67	2
4	9	83	1	7	2	69	1
5	7	85	2	10	1	70	1
6	2	88	2	11	2	72	1
7	2	90	2	12	1	73	1
8	4	91	1	13	1	78	2
9	1	93	1	15	1	81	1
10	2	95	1	17	1	82	3
12	2	96	2	18	1	85	2
13	1	97	1	19	1	88	1
14	1	99	1	20	2	90	1
17	3	100	3	21	2	91	1
18	3	101	3	22	4	93	2
21	1	103	1	23	3	98	1
23	1	104	1	24	2	102	1
27	1	105	1	25	2	103	1
29	1	108	2	26	1	107	1
35	1	109	2	27	2	110	1
		110	1	28	1	140	1
		114	1	29	3		

VAL. FRQ.

CASE 60

0	18
1	10
2	7
3	6
4	5
5	4
6	1
7	3
8	2
9	4
10	1
11	5
12	1
13	1
14	2
15	1
16	4
17	1
18	2
19	2
20	2
21	2
22	2
25	1
26	3
28	1
30	2
31	2
32	1
38	1
50	1
53	1
90	1

APPENDIX C

COMPUTER PROGRAMS

Computer programs, to prepare Table (4.1) and Table (4.2) (Computer program 1), to find the estimates for the examples one through three of chapter three (computer program 2), and to generate the data for the sixty cases of the example four of chapter three and to find the estimates for them (computer program 3), are given in the following pages. There are two subroutines, namely GOLDEN (Golden section search), SVS (Single variable search), and one function, namely FCN (to evaluate efficiency function), which are common to the three programs mentioned above. These are given at the beginning under the title COMMON SUBROUTINES. After that the three programs are given.

One who wants to use any one of the programs, must include the common subroutines after the main program of the computer program he chooses.

COMMON SUBROUTINES

```
SUBROUTINE SVS(FCN,TNOT)
IMPLICIT REAL*8 (A-H,O-Z)
COMMON P,R,DETLAM
Q=1.0DO-P
UPPER=1.0DO/DABS(Q)
ALOW=-1.0DO/DSQRT(DABS(Q))
RIOU=0.000001D0
START=0.9999
U=START
S=0.0001D0
A=FCN(U)
U1=U+S
B=FCN(U1)
IF (B.GT.A) GO TO 5
IF (B.EQ.A) GO TO 6
M=3
1  U2=START+S*2.0D0**(M-2)
   IF (U2.GE.UPPER) GO TO 99
   A=B
   B=FCN(U2)
   IF (A.EQ.B) GO TO 7
   IF (A.LT.B) GO TO 8
   M=M+1
   U=U1
   U1=U2
   GO TO 1
7  CALL GOLDEN(FCN,U1,U2,RIOU,50,TNOT)
   GO TO 999
99  CALL GOLDEN(FCN,U1,UPPER,RIOU,50,TNOT)
   GO TO 999
8  CALL GOLDEN(FCN,U,U2,RIOU,50,TNOT)
   GO TO 999
6  M=2
   CALL GOLDEN(FCN,U,U1,RIOU,50,TNOT)
   GO TO 999
5  M=3
2  U2=START-S*2.0D0**(M-3)
   IF (U2.LE.ALOW) GO TO 77
   B=A
   A=FCN(U2)
   IF (A.EQ.B) GO TO 17
   IF (A.GT.B) GO TO 18
   U1=U
   U=U2
   M=M+1
   GO TO 2
17  CALL GOLDEN(FCN,U2,U,RIOU,50,TNOT)
   GO TO 999
18  CALL GOLDEN(FCN,U2,U1,RIOU,50,TNOT)
   GO TO 999
77  CALL GOLDEN(FCN,ALOW,U,RIOU,50,TNOT)
999 RETURN
END
```



```

SUBROUTINE GOLDEN (FCN, ENDL, ENDR, RIOU, NMAX, XOLD)
IMPLICIT REAL*8 (A-H, O-Z)
COMMON P, R, DETLAM
SIGMA= (3.0D0-DSQRT(5.0D0))/2.0D0
AL=ENDR-ENDL
XOLD=ENDL+SIGMA*AL
FOLD=FCN(XOLD)
DO 1 I=2, NMAX
XNEW=ENDL+ENDR-XOLD
FNEW=FCN(XNEW)
IF (FNEW.LE.FOLD) GO TO 5
IF (XNEW.LT.XOLD) GO TO 6
ENDR=XNEW
20 AL=ENDR-ENDL
GO TO 30
6 ENDL=XNEW
GO TO 20
5 IF (XNEW.LT.XOLD) GO TO 7
ENDL=XOLD
8 XOLD=XNEW
FOLD=FNEW
GO TO 20
7 ENDR=XOLD
GO TO 8
30 IF (AL.LE.RIOU) GO TO 40
1 CONTINUE
40 RETURN
END

```

```

FUNCTION FCN(X)
IMPLICIT REAL *8 (A-H, O-Z)
COMMON P, R, DETLAM
ZR = 1.0D-73
Q=1.0D0-P
A=R*P** (R+2.DO)/Q
B=1.0D0-Q*X
C=(DABS((B*DLOG(B/P)/P-Q*(1.0D0-X)/P))**2.D0
D=B** (2.D0*R+2.0D0)/(1.0D0-Q*DABS(X)**2.D0)**R
E=P**R*B**2.D0+R*Q*P**R*(DABS(1.0D0-X))**2.D0
G=-A*C/DETLAM
F=D-E
IF (DABS(F).LT.ZR) GO TO 5
FCN=G/F
GO TO 7
5 FCN=-1.0D0
7 RETURN
END

```


COMPUTER PROGRAM 1

```

IMPLICIT REAL*8 (A-H,O-Z)
EXTERNAL FCN
DIMENSION PR(14),RR(26),EFF(26,14),TZERO(26,14),TTEST(52),EFF1(26,
114),EFF2(26,14)
COMMON P,R,DETLAM
DATA EFF,TZERO/728*0.0D0/
DATA EFF1,EFF2/728*0.0D0/
DATA PR/.05D0,.06D0,.07D0,.08D0,.09D0,.1D0,.2D0,.3D0,.4D0,.5D0,.6D
10,.7D0,.8D0,.9D0/
DATA RR/.5D0,1.0D0,1.5D0,2.0D0,2.5D0,3.0D0,3.5D0,4.0D0,4.5D0,5.0D0,6.0D
1,7.0D0,8.0D0,9.0D0,10.0D0,11.0D0,12.0D0,13.0D0,14.0D0,15.0D0,16.0D0,17.0D0,18
1.0D0,19.0D0,0.0D0,0.0D0/
DATA TTEST/52*0.0D0/
WRITE (6,444)
444  FORMAT('1')
EPS=0.0001D0
DO 1 J=1,14
P=PR(J)
Q=1.0D0-P
K=1
7  TEST=Q**(K+1)/(P*(DFLOAT(K)+2.0D0))
K=K+1
GO TO 7
8  M=K
WRITE (6,21) M
21  FORMAT(15X,'UPPER BOUND=',I8)
DO 2 I=1,24
R=RR(I)
DETLAM=0.0D0
DO 10 N=1,M
RATION=1.0D0
DO 12 JJ=1,N
12  RATION=RATION*DFLOAT(JJ)/(DFLOAT(JJ)+R)
DETLAM=DETLAM+Q**N*RATION/DFLOAT(N+1)
10  CONTINUE
CALL SVS(FCN,TNOT)
TZERO(I,J)=TNOT
TPL=DEXP(0.5D0*DLOG(TNOT))
TMIN=DEXP(1.5D0*DLOG(TNOT))
EFF2(I,J)=-FCN(TPL)
EFF1(I,J)=-FCN(TMIN)
EFF(I,J)=-FCN(TNOT)
2  CONTINUE
1  CONTINUE
WRITE (6,200) (PR(I),I=1,14)
200  FORMAT(///,50X,'P  VALUES',/,12X,14(2X,F5.3,1X),/,12X,14(2X,'-----
1',1X),/)
DO 13 I=1,24
WRITE (6,202) (EFF1(I,J),J=1,14)
WRITE (6,201) RR(I),(EFF(I,J),J=1,14)
WRITE (6,202) (EFF2(I,J),J=1,14)

```



```
201  FORMAT(3X,'R=',F5.2,2X,14(2X,F5.3,1X))  
    WRITE (6,203) (TZERO(I,J),J=1,14)  
202  FORMAT(12X,14(1X,'(',F5.3,')'))  
203  FORMAT(5X,'OP.TNOT',14(2X,F5.3,1X),//)  
13   CONTINUE  
    STOP  
    END
```



```

IMPLICIT REAL*8 (A-H,O-Z)
EXTERNAL FCN
DIMENSION FRQ(20),OBS(20)
COMMON P,R,DETLAM
WRITE (6,444)
444  FORMAT('1')
EPS=0.0001D0
WRITE (6,91)
91  FORMAT(5X,'READ IN NO. OF OBS. AND VARIATES IN 215 FORMAT',/)
READ (5,81) NOBS, NVAR
81  FORMAT(2I5)
WRITE (6,92)
92  FORMAT(5X,'NOW GIVE ME VARIATE VALUES AND CORRESPONDING FREQUENCY
1IN THE FORMAT 2F10.5 FOR EACH CARD',/)
SUM=0.0D0
SUM1=0.0D0
DO 4 I=1,NVAR
READ (5,82) CBS(I),FRQ(I)
82  FORMAT(2F10.5)
SUM=SUM+OBS(I)*FRQ(I)
SUM1=SUM1+(OBS(I)**2.0D0)*FRQ(I)
XBAR=SUM/DFLOAT(NOBS)
SSQR=(SUM1-DFLOAT(NOBS)*XBAR**2.0D0)/DFLOAT(NOBS-1)
WRITE (6,209) XBAR, SSQR
209  FORMAT(15X,'USING GIVEN DATA FOLLOWING INFORMATION IS OBTAINED',//
1,20X,'SAMPLE MEAN      =',F7.4, '//,20X,'SAMPLE VARIANCE  =',F7.4, '//
2)
P=XBAR/SSQR
R=XBAR**2.0D0/(SSQR-XBAR)
Q=1.0D0-P
K=1
7  TEST=Q**(K+1)/(DFLOAT(K+2))
IF (TEST.LE.EPS) GO TO 8
K=K+1
GO TO 7
8  M=K
DETLAM=0.0D0
DO 10 N=1,M
RATTON=1.0D0
DO 12 JJ=1,N
12  RATTON=RATTON*DFLOAT(JJ)/(DFLOAT(JJ)+R)
DETLAM=DETLAM+Q**N*RATTON/DFLOAT(N+1)
10  CONTINUE
L=1
95  CALL SVS (FCN,TNOT)
EFF1=-FCN(TNOT)
RSTAR=R
PSTAR=P
SUM2=0.0D0
DO 14 J=1,NVAR
14  SUM2=SUM2+FRQ(J)*TNOT**OBS(J)
TXBAR=SUM2/DFLOAT(NOBS)

```



```

CALL COMPR(TNOT, TXBAR, XBAR, RSTAR)
P=R/(R+XBAR)
Q=1.0D0-P
K=1
77 TEST=Q*(K+1)/(P*DFLOAT(K+2))
   IF (TEST.LE.EPS) GO TO 88
   K=K+1
   GO TO 77
88 M=K
   DETLAM=0.0D0
   DO 51 N=1,M
   RATION=1.0D0
   DO 52 JJ=1,N
52 RATION=RATION*DFLOAT(JJ)/(DFLOAT(JJ)+R)
51 DETLAM=DETLAM+Q**N*RATION/DFLOAT(N+1)
   EFF2=-FCN(TNOT)
   IF (DABS(P-PSTAR).GT.0.005) GO TO 29
   IF (DABS(R-RSTAR).GT.0.005) GO TO 29
   IF (DABS(EFF1-EFF2).GT.0.04) GO TO 29
   WRITE (6,94) P,R,EFF2,TNOT,L
   FORMAT(20X,'ESTIMATE OF P      =' ,F7.5,/,/,20X,'ESTIMATE OF R      =' ,F
17.4,/,/,20X,'EFFICIENCY          =' ,F7.5,/,/,20X,'OPTIMUM TNOT      =' ,F
27.5,/,/,20X,'NO.OF ITERATIONS =' ,F14,/,/)
   GO TO 999
29 L=L+1
   GO TO 95
999 STOP
   END

```

```

SUBROUTINE COMPR(T, TX, XB, RST)
IMPLICIT REAL*8 (A-H, O-Z)
COMMON P, R, DETLAM
R1=RST
30 B=R1+XB*(1.0D0-T)
   TY=(R1/B)**R1
   F=TX-TY
   FP1=-TY
   FP2=(XB*(1.0D0-T)/B)+DLOG(R1/B)
   FP=FP1*FP2
   R2=R1-F/FP
   IF (DABS(R2-R1).LE.0.00001D0) GO TO 66
   R1=R2
   GO TO 30
66 R=R2
   RETURN
   END

```



```

IMPLICIT REAL*8 (A-H,O-Z)
EXTERNAL FCN
DIMENSION FRQ(200),OBS(200),PVECT(10),RVECT(10),NVECT(10),DAT(200)
1,ISD(200)
COMMON P,R,DETLAM
DATA RVECT/5.0D0,2.5D0,0.5D0,7*0.0D0/
DATA PVECT/.8D0,.5D0,.3D0,.1D0,.05D0,5*0.0D0/
DATA NVECT/15,30,50,100,6*0/
CALL OVFLOW
WRITE(5,541)
541. FORMAT(/,15X,'READ ISEED USED IN GENERATING DATA')
READ(5,978) IS
978 FORMAT(I10)
WRITE(6,979) IS
979 FORMAT(15X,'ISEED USED TO GENERATE DATA=',I10)
CALL INT(IS,ISD,200)
WRITE(6,444)
444 FORMAT('1')
EPS=0.0001D0
NCASE=0
DO 112 IN=1,4
DO 113 IP=1,5
DO 114 IR=1,3
P=PVECT(IP)
R=RVECT(IR)
NOBS=NVECT(IN)
NCASE=NCASE+1
NCP=2*NCASE
NCM=NCP-1
ISEED1=ISD(NCP)
ISEED2=ISD(NCM)
CALL GNRNB(NOBS,OBS,P,R,ISEED2,ISDEED1)
OBMX=-99999.D0
DO 321 LK=1,NOBS
IF (OBS(LK).GT.OBMX) OBMX=OBS(LK)
321 CONTINUE
OBLM=OBMX+10.D0
NVAR=0
390 OBMN=OBMX+5.D0
DO 370 MI=1,NOBS
IF (OBS(MI).LT.OBMN) OBMN=OBS(MI)
370 CONTINUE
ICNT=0
DO 380 MJ=1,NOBS
IF (OBS(MJ).NE.OBMN) GO TO 380
ICNT=ICNT+1
OBS(MJ)=OBLM
380 CONTINUE
NVAR=NVAR+1
DAT(NVAR)=OBMN
FRQ(NVAR)=DFLOAT(ICNT)
IF (OBMN.EQ.OBMX) GO TO 395

```



```

      GO TO 390
395  WRITE (6,889)
889  FORMAT(//,10X,89('*'),/,40X,'DATA GENERATED IS GIVEN BELOW',/,39X,
131('*'),/)
      WRITE (6,301) (DAT(L0),L0=1,NVAR)
301  FORMAT(20X,'VALUES',/,20X,6('-'),/, (5X,20(1X,F5.0),/))
      WRITE (6,302) (FRQ(L0),L0=1,NVAR)
302  format920X,'FREQUENCIES',/,20X,11('-'),/, (5X,20(1X,F5.0),/))
      WRITE(6,543)
543  FORMAT(//,10X,'DATA WRITTEN ABOVE IS BASED ON FOLLOWING PARAMETERS
1',/,10X,52('-'))
      WRITE (6,544) NCASE,P,R,NOBS,NVAR
544  FORMAT(/,10X,'CASE NO.=',I5.5X,'P=',F7.4,5X,'R=',F7.2,5X,'N=',I5
1X,'NO.OF VALUES=',I3,/)
      SUM=0.0D0
      SUM1=0.0D0
      DO 445 I=1,NVAR
      SUM=SUM+FRQ(I)*DAT(I)
445  SUM1=SUM1+FRQ(I)*DAT(I)**2.0D0
      XBAR=SUM/DFLOAT(NOBS)
      SSQR=(SUM1-DFLOAT(NOBS)*XBAR**2.0D0)/DFLOAT(NOBS-1)
      IF (XBAR.GE.SSQR) GO TO 71
      P=XBAR/SSQR
      R=XBAR**2.0D0/(SSQR-XBAR)
      GO TO 72
71  R=5.0D0
      P=0.9D0
      GO TO 114
72  Q=1.0D0-P
      K=1
7  TEST=Q**(K+1)/(DFLOAT(K+2))
      IF (TEST.LE.EPS) GO TO 8
      K=K+1
      GO TO 7
8  M=K
      DETLAM=0.0D0
      DO 10 N=1,M
      RATION=1.0D0
      DO 12 JJ=1,N
12  RATION=RATION*DFLOAT(JJ)/(DFLOAT(JJ)+R)
      DETLAM=DETLAM+Q**N*RATION/DFLOAT(N+1)
10  CONTINUE
      L=1
95  CALL SVS (FCN,TNOT)
      EFF1=-FCN (TNOT)
      RSTAR=R
      PSTAR=P
      SUM2=0.0D0
      DO 145 J=1,NVAR
145  SUM2=SUM2+FRQ(J) (TNOT**DAT(J))
      TXBAR=SUM2/DFLOAT(NOBS)
      CALL COMPR (TNOT, TXBAR, XBAR, RSTAR)

```



```

P=R/(R+XBAR)
Q=1.0D0-P
K=1
77  TEST=Q**(K+1)/(P*DFLOAT(K+2))
    IF (TEST.LE.EPS) GO TO 88
    K=K+1
    GO TO 77
88  M=K
    DETLAM=0.0D0
    DO 51 N=1,M
    RATION=1.0D0
    DO 52 JJ=1,N
52  RATION=RATION*DFLOAT(JJ)/(DFLOAT(JJ)+R)
51  DETLAM=DETLAM+Q**N*RATION/DFLOAT(N+1)
    EFF2=-FCN(TNOT)
    IF (L.NE.1) GO TO 141
    UNOT=-DLOG(DABS(TNOT))
    WRITE(6,98) PSTAR,RSTAR,P,R,EFF1,EFF2,UNOT
98  FORMAT(/,11X,57('&'),/,15X,'VALUES OBTAINED AT THE END OF FIRST IT
    LERATION',/,12X,'PTHIL',3X,'RTHIL',3X,'PSTAR',3X,'RSTAR',3X,'ETHIL'
    2,3X,'ESTAR',3X,'UNOT',/,12X,F5.3,2X,F7.3,2X,F5.3,2X,F7.3,2X,F5.3,3
    3X,F5.3,2X,F7.4,/,11X,57('&'),/)
141 IF (DABS(P-PSTAR).GT.0.001) GO TO 29
    IF (DABS(R-RSTAR).GT.0.005) GO TO 29
    IF (DABS(EFF1-EFF2).GT.0.01) GO TO 29
    UNOT=-DLOG(DABS(TNOT))
    WRITE(6,94) XBAR,SSQR,P,R,EFF2,UNOT,L
94  FORMAT(15X,'USING THE ABOVE DATA FOLLOWING INFORMATION IS OBTAINED
    1/,/,15X,54('-'),/,10X,'SPLE.MEAN',2X,'SPLE.VAR',3X,'EST.OF P',3X,'
    2EST.OF R',6X,'EFF',6X,'OP.UNOT',3X,'NO.OF ITR',/,10X,7(9('-'),2X),
    3/,10X,F9.3,2X,F9.3,2X,F9.3,2X,F9.3,2X,F9.3,2X,F9.4,5X,I3,/)
    GO TO 114
29  L=L+1
    GO TO 95
114 CONTINUE
113 CONTINUE
112 CONTINUE
    STOP
    END

```



```

SUBROUTINE GNRNB(N,OBS,P,R,ISEED2,ISEED1)
REAL*8 P,R,OBS
DIMENSION OBS(200),GAM(200),K(1)
P1=SNGL(P)
R1=SNGL(R)
CALL GAMA(R1,ISEED1,GAM,N)
DO 1 I=1,N
  R1AM=GAM(I)*(1.-P1)/P1
  CALL GGPOSH(R1AM,ISEED2,1,K,IER)
1 OBS(I)=DBLE(FLOAT(K(1)))
RETURN
END

```

```

SUBROUTINE COMPR(T,TX,XB,RST)
IMPLICIT REAL*8 (A-H,O-Z)
COMMON P,R,DETLAM
R1=RST
30 B=R1+XB*(1.0D0-T)
  TY=DABS(R1/B)**R1
  F=TY-TX
  FP1=TY
  FP2=(XB*(1.0D0-T)/B)+DLOG(DABS(R1/B))
  FP=FP1*FP2
  R2=R1-F/FP
  IF(DABS(R2-R1).LE.0.0001D0) GO TO 66
  R1=R2
  GO TO 30
66 R=R2
RETURN
END

```


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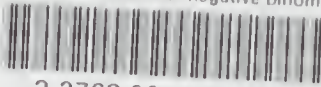
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